Determination of Rayleigh Phase Velocity of Soil Using Rayleigh Wave Equation

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ABSTRACT: Wave propagation and related ground vibration are significant areas of study in wide scientific disciplines such as seismology, geology, geotechnical engineering, geophysics etc. The main reason for the importance of research in this area is that the wave propagation in soil medium and related ground movements detrimentally affect the structures founded on it. Rayleigh wave is an important surface wave which exists in homogeneous elastic half space. In this study a review of different analytical solutions of Rayleigh wave equation is given and real parts of the roots are compared. Also the root of the Raleigh wave equation is obtained by optimization using MS Excel. Optimization technique gives comparable values for the real part of the root with available analytical solutions for the entire range of Poisson ratio from 0 to 0.5.

Keywords: Wave propagation, ground vibrations, Rayleigh wave

1. Introduction
Wave propagation in soil medium due to natural cause is mainly attributed to earthquakes. Wave propagation also happens due to anthropogenic sources such as pile driving, traffic, blasting, moving parts of machinery etc. Generally waves can be classified into two types body waves and surface waves. Body waves travelling through the interior of the earth are of two types p-waves (Primary waves) and s-waves (shear waves). Surface waves result from the interaction between the body waves and surface of the earth.

Rayleigh waves produced by interaction of P-waves and SV (Shear – vertical) waves with earth surface involve both horizontal and vertical particle motion. Rayleigh waves propagate along the ground surface. These waves are created during earthquakes in addition to body waves and carry major share of the energy dissipated creating destruction of structures founded on the soil. (Kramer 1996)

2. Motivation and Objective
The condition of propagation and the existence of Rayleigh waves were predicted theoretically by Lord Rayleigh. He proposed the characteristic Rayleigh wave equation for a homogeneous and isotropic half-space with free boundary conditions. The solutions of the secular Rayleigh wave equation give the velocity of the waves in medium. In a homogeneous infinite half space Rayleigh wave is non-dispersive in nature. Main objective of the study is to review the different analytical solutions of Rayleigh wave equation to obtain the Rayleigh wave velocity. The study also intends to find the root of Rayleigh equation using optimization technique and to compare the real part of the root obtained using optimization technique with available analytical solutions.

3. Methodology
The well-known Rayleigh wave equation is given by
\[ x^3 - 8x^2 + 8x(3-2\gamma) - 16(1-\gamma) = 0 \] (1)

Where \( x = (c/\beta)^2 \)
\( \gamma \) -velocity ratio, \( c \) - Rayleigh wave velocity, \( \beta \)-shear wave velocity

The Rayleigh wave equation can be solved using different methods

3.1 Malischewsky Solution
Malischewsky (2000) solved this secular equation using the theory of cubic equations and auxiliary functions. The solution is given by
\[ x(\gamma) = \frac{2}{3} \left[ 4 - \frac{1}{h_3(\gamma)} + \text{sign}[h_4(\gamma)] \cdot \sqrt{\text{sign}[h_4(\gamma)] \cdot h_2(\gamma)} \right] \] (2)

Where auxiliary functions are given by
\[ h_3(\gamma) = 3\sqrt{33 - 186\gamma + 321\gamma^2 - 192\gamma^3} \] (3)
\[ h_4(\gamma) = -17 + 45\gamma + h_1(\gamma) \] (4)
\[ h_5(\gamma) = 17 - 45\gamma + h_1(\gamma) \] (5)
\[ h_6(\gamma) = 1/6 - \gamma \] (6)

And
\[ \gamma = \frac{1 - 2v}{2(1-v)} \] (7)

where \( v \) – Poisson ratio, \( \alpha \) – primary wave velocity

When this solution was obtained using Poisson’s ratio varying from 0 to 0.5, it was found out that the real part of the solution exists only in the range of 0.26 ≤ \( v \) ≤ 0.50. The Poisson ratio of value 0.26 is called critical Poisson ratio. The value of root of Rayleigh equation was obtained as in the range of 0.85 to 0.91 as the Poisson ratio was varied from 0.26 to 0.50.

The Rayleigh speed is given by,
\[ c = \sqrt{x(\gamma)} \cdot \beta \] (8)

3.2 Using Cardon’s Formula and Maple’s Procedure
Mechkour (2003) solved the Rayleigh wave equation using Cardon’s formula and taking advantage of Maple’s Procedure
The solution is given as

\[ \theta_1 = \frac{2}{3} \sqrt[3]{-17 + 45\alpha + 3\sqrt{33 - 186\alpha + 321\alpha^2 - 192\alpha^3}} \]
\[ - \frac{3}{2} \sqrt[3]{-17 + 45\alpha + 3\sqrt{33 - 186\alpha + 321\alpha^2 - 192\alpha^3}} + 8 \]
\[ \frac{3}{2} \sqrt[3]{-17 + 45\alpha + 3\sqrt{33 - 186\alpha + 321\alpha^2 - 192\alpha^3}} \]  

(9)

The phase velocity \( c \) of Rayleigh waves is obtained as

\[ c = \frac{e \theta_1}{\theta} \]  

(10)

When this solution was obtained using Poisson’s ratio varying from 0 to 0.5, it was found out that the real part of the solution exists only in the range of \( 0.26 \leq \nu \leq 0.5 \).

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### 3.3 Using the theory of Reiman’s problem

Nkemzi (2007) using the theory of Reiman’s problem obtained the Rayleigh velocity

\[ t^3 - 8t^2 + (24 - 16\nu)t - 16(1 - \gamma) = 0 \]  

(11)

The solution in terms of \( m_0 \) is given by

\[ m_0 = \frac{2}{3} \omega \sqrt[3]{17 - 45\nu + 3\sqrt{33 - 186\nu + 321\nu^2 - 192\nu^3}} \]
\[ - \frac{3}{2} \omega \sqrt[3]{17 - 45\nu + 3\sqrt{33 - 186\nu + 321\nu^2 - 192\nu^3}} + \frac{1}{4 \omega \sqrt{3}} \]
\[ - \frac{1}{4 \omega \sqrt{3}} \]  

(12)

Also Nkemzi provided direct method to find out Rayleigh speed

\[ v_0^2 = \frac{8}{3} - \frac{2}{3} f(\nu) + \frac{3}{2} \omega - \frac{16\nu}{3} / f(\nu) \]  

(13)

When this solution was obtained using Poisson’s ratio varying from 0 to 0.5, it was found out that the real part of the solution exists only in the range of \( 0.26 \leq \nu \leq 0.5 \). Also the solution gives values less than one only for values of Poisson ratio equals to 0.26. For all other values of Poisson ratio, the root of the equation is greater than one.

### 3.4 Using partial fractions

Rahman and Barber Solution (1995) using partial fractions found out the solution to Rayleigh wave equation. Unlike the previous methods, this solution was found out for three ranges of Poisson Ratio of material i.e. \( 0 \leq \nu < 0.26 \), \( \nu = 0.26 \) and \( 0.26 < \nu \leq 0.5 \).

#### Case 1

\( 0 \leq \nu < 0.26 \)

\[ m_1 = \frac{8}{3} + 2 \sqrt[3]{\frac{e}{3} \cos \left( \phi + \frac{2\pi}{3} \right)} \]  

(14)

\[ m_2 = \frac{8}{3} + 2 \sqrt[3]{\frac{e}{3} \cos \left( \phi + \frac{4\pi}{3} \right)} \]  

(15)

Where \( m_1 \), \( m_2 \) and \( m_3 \) are the three real roots. Out of these three roots only \( m_1 \) satisfies the physical condition that Rayleigh velocity is less than shear velocity.

Here

\[ p = \frac{8}{3}(1 - 6\nu) \]  

(17)

\[ q = \frac{16}{27}(17 - 45\nu) \]  

(18)

\[ \alpha = \frac{8}{3} \frac{(1 - 2\nu)}{2(1 - \nu)} \]  

(19)

The value of \( m_1 \) varies from 0.56 to 0.92 as Poisson ratio varies from 0 to 0.25.

#### Case 2

\( \nu = 0.26 \)

Here also there are three real roots. Roots \( m_2 \) and \( m_3 \) are equal and their values are incompatible with physical condition.

\[ m_1 = \frac{8}{3} + 2\sigma \]  

(20)

\[ m_2 = m_3 = \frac{8}{3} - \sigma \]  

(21)

\[ \sigma = \frac{2 + 2(11 - 56\nu)}{3 \sqrt{2(1 - \nu)^2}} \]  

(22)

The value of \( m_1 \) is 0.85 for Poisson ratio equals to critical value.

#### Case 3

\( 0.26 < \nu \leq 0.5 \)

Here there is only real root \( m_1 \).

\[ m_1 = \frac{8}{3} + \gamma + \eta \]  

(23)

\[ \gamma = \frac{2}{3} \sqrt[3]{-17 + 45\nu + \sqrt{(17 - 45\nu)^2 + 8(1 - 6\nu)^3}} \]  

(24)

\[ \eta = \frac{2}{3} \sqrt[3]{-17 + 45\nu - \sqrt{(17 - 45\nu)^2 + 8(1 - 6\nu)^3}} \]  

(25)

The value of \( m_1 \) varies from 0.86 to 0.91 as Poisson ratio varies from critical value to 0.50.

### 3.5 Optimization technique

Optimization is effectively used to solve problems in different areas in engineering. In these problems using limited resources an optimal way is found out to achieve the objective of the situation. This may be maximizing or minimizing certain objective functions. For using optimization technique for the given problem, a mathematical model is formulated to represent the situation. The model consists of following components such as decision variable, objective function and constraints. Decision variables represent unknown quantities. Objective function is a mathematical expression of the problem in decision variables.
Constraints are the limitations or requirements of the problem are expressed as inequalities or equations in decision variables. The various techniques used in optimization are classical method such as analytical methods using differential calculus and numerical methods. Also advanced methods using genetic algorithms, ant colony optimization etc can be effectively used for optimization problems. Finding out the roots of Rayleigh equation can be solved as an optimization problem. Here the objective function is the Rayleigh secular equation which has to be minimized.

Microsoft Excel can be effectively used for find roots of higher order equations with single or multiple variables using optimization. Here Excel includes an optimisation tool called Solver1. Solver uses a complex version of Newton’s method, which combines a quasi-Newton algorithm with a conjugate gradient.

4. Comparison of results

The table 1 shows the values for root of real root of the Rayleigh secular equation solved using different methods. It can be seen that only Rahman and Barber method and Optimization method gives real roots for the entire spectrum of Poisson ratio for 0 to 0.50. Both the methods give same values for Poisson ratio 0.26≤ ν <0.50. For Poisson ratio 0≤ν<0.26 the values for the root are different, but comparable. The other three methods Malischewsky, Nkemzi and Mechkour give complex roots for 0≤ν<0.26. Malischewsky, and Houri give same value for the root as that of optimisation method for Poisson ratio 0.26≤ ν <0.50. But Nkemzi values are different for this range.

Table -1 Comparison of root of Rayleigh Equation

<table>
<thead>
<tr>
<th>Poisson Ratio (ν)</th>
<th>Velocity ratio (γ)</th>
<th>Root of Rayleigh Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Malischewsky</td>
<td>Nkemzi</td>
</tr>
<tr>
<td>0</td>
<td>0.50</td>
<td>compl ex</td>
</tr>
<tr>
<td>0.05</td>
<td>0.47</td>
<td>compl ex</td>
</tr>
<tr>
<td>0.10</td>
<td>0.44</td>
<td>compl ex</td>
</tr>
<tr>
<td>0.15</td>
<td>0.41</td>
<td>compl ex</td>
</tr>
<tr>
<td>0.2</td>
<td>0.38</td>
<td>compl ex</td>
</tr>
<tr>
<td>0.25</td>
<td>0.33</td>
<td>compl ex</td>
</tr>
<tr>
<td>0.26</td>
<td>-3.73</td>
<td>compl ex</td>
</tr>
<tr>
<td>0.3</td>
<td>0.29</td>
<td>0.86 1.00</td>
</tr>
<tr>
<td>0.35</td>
<td>0.23</td>
<td>0.87 1.20</td>
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<td>0.09</td>
<td>0.90 1.89</td>
</tr>
<tr>
<td>0.5</td>
<td>0.00</td>
<td>0.91 3.09</td>
</tr>
</tbody>
</table>

The relation between Rayleigh velocity and shear velocity is given by Rayleigh phase velocity, \( c = \beta \sqrt{x} \) where \( x \) is the root of the Rayleigh equation.

Fig. 1 Variation of \( \sqrt{x} \) with Poisson ratio

The figure 1 shows the variation of \( \sqrt{x} \) with Poisson ratio. It is clear that Rayleigh velocity in all the cases is slightly slower than shear wave velocity. For Rahman and Barber solution the Rayleigh velocity will be 0.75 times shear wave velocity to 0.96 times shear wave velocity as Poisson ratio varies from 0 to 0.50. For solution using optimization the Rayleigh velocity will be 0.87 times shear wave velocity to 0.96 times shear wave velocity as Poisson ratio varies from 0 to 0.50. For Poisson ratio 0.26≤ ν <0.50, Malischewsky and Mechkour give same value for the root as that of optimisation method. The values for the root of Rayleigh equation given by Nkemzi are higher one for except Poisson ratio equals to 0.26. The formulation of Rayleigh wave equation is based on theory of elasticity and all these analytical solutions hold good in the case wave propagation through soil at low strain level.

5. Conclusions

The Rahman and Barber method and Optimization technique give the root of the Rayleigh secular equation for the entire range of Poisson ratio varying from 0 to 0.5.

Using Rahman and Barber method the root of Rayleigh equation increases to maximum value (0.92) and then decreases to 0.85 at critical value. The value of the root again increases after the critical value of Poisson ratio.

Malischewsky and Mechkour methods can be used for value of Poisson Ratio greater than critical value. Also Mechkour Method gives complex value for the root for Poisson ratio equal to 0.40. Using Nkemzi method, it was found out that the real part of the solution exists for Poisson ratio greater than critical value and the solution
gives values less than one only for Poisson ratio equals to 0.26.

The solution to Rayleigh secular wave equation using optimization technique indicates the root of Rayleigh equation increases from 0.76 to 0.91 as Poisson ratio varies from 0 to 0.50. The Rayleigh velocity will be 0.87 times shear wave velocity to 0.96 times shear wave velocity as Poisson ratio varies from 0 to 0.50.

References


